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Oscillation criteria for certain second-order Emden-Fowler delay functional dynamic equations with damping on time scales

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Abstract

This paper is concerned with oscillations of certain second-order Emden-Fowler variable delay functional dynamic equations with damping and neutral of the form

$$\left[A(t)\varphi\left(y^{\Delta}(t)\right)\right]^{\Delta}+b(t)\varphi\left(y^{\Delta}(t)\right)+P(t)F\left(\varphi\left(x\left(\delta(t)\right)\right)\right)-Q(t)f\left(\varphi\left(x\left(\gamma(t)\right)\right)\right)=0$$

on an arbitrary time scale $\mathbf{T}$, where $y(t)=x(t)+B(t)x(\tau(t))$ and $\varphi(u)=|u|^{\lambda-1}u$ ($\lambda>0$). By using the generalized Riccati transformation and the inequality technique, some new oscillation criteria for the equations are established. Our results extend and improve some known results, but they also unify the oscillation of second-order Emden-Fowler delay differential equations with damping and second-order Emden-Fowler delay difference equations with damping. Examples are given to illustrate the importance of our results.

MSC: 34K11; 34C10; 39A10

Keywords: oscillation; delay dynamic equations; Riccati transformation; damping

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced in [1], in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory; see [2–6]. A time scale $\mathbf{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of discrete dynamic models of populations which are in season (and may follow a difference scheme with variable step size but are often modeled by continuous dynamic systems), die out, say in winter, while their eggs are incubating or dormant, and then in season again, and hatching gives rise to a nonoverlapping population (see [4]). Not only does the new theory of so-called 'dynamic equations' unify the theories of differential equations and difference equations, but also it extends these classical cases to cases 'in between', *e.g.*, to so-called q-difference equations when $\mathbf{T}=q^{\mathbf{N}_0}=\{q^t:t\in\mathbf{N}_0\}$ for some $q>1$ (which has important applications in quantum theory) and can be applied on different time scales like $\mathbf{T}=h\mathbf{N}$, $\mathbf{T}=\mathbf{N}^2=\{t^2:t\in\mathbf{N}\}$, $\mathbf{T}=\mathbf{T}_n=\{t_n=\sum_{k=1}^{n}\frac{1}{k},n\in\mathbf{N}_0\}$ and the space of the harmonic numbers.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the studies in [7–9]. However, there are few results dealing with the oscillation of the solutions of second-order delay dynamic equations with damping on time scales [10–28]. In this article, we study oscillatory behavior of all solutions of second-order Emden-Fowler variable delay functional dynamic equations with damping and neutral

$$\begin{aligned}&\bigl[A(t)\varphi\bigl(y^{\Delta}(t)\bigr)\bigr]^{\Delta}+b(t)\varphi\bigl(y^{\Delta}(t)\bigr)+P(t)F\bigl(\varphi\bigl(x\bigl(\delta(t)\bigr)\bigr)\bigr)-Q(t)f\bigl(\varphi\bigl(x\bigl(\gamma(t)\bigr)\bigr)\bigr)=0,\\&\quad t\in\mathbf{T},t\geq t_0,\end{aligned}\tag{1.1}$$

where $y(t)=x(t)+B(t)x(\tau(t))$, $\varphi(u)=|u|^{\lambda-1}u$, $\lambda>0$, subject to the following hypotheses:

- (H_1) $\mathbf{T}$ is a time scale (*i.e.*, a nonempty closed subset of the real numbers $\mathbf{R}$) which is unbounded above, and $t_0\in\mathbf{T}$ with $t_0>0$, we define a time scale interval of the form $[t_0,+\infty)_{\mathbf{T}}$ by $[t_0,+\infty)_{\mathbf{T}}=[t_0,+\infty)\cap\mathbf{T}$. $A(t),B(t),b(t),P(t),Q(t)\in C_{rd}(\mathbf{T},\mathbf{R})$, *i.e.*, $A(t),B(t),b(t),P(t),Q(t):\mathbf{T}\to\mathbf{R}$ are rd-continuous functions. $F(u),f(u):\mathbf{R}\to\mathbf{R}$ are continuous functions with $uF(u)>0$ ($u\neq0$) and $uf(u)>0$ ($u\neq0$).
- (H_2) $\tau(t),\delta(t),\gamma(t):\mathbf{T}\to\mathbf{T}$ are delay functions such that $\tau(t)\leq t$, $\lim_{t\to+\infty}\tau(t)=+\infty$; $\gamma(t)=\delta(t)\leq t$, $\lim_{t\to+\infty}\delta(t)=+\infty$.
- (H_3) $0\leq B(t)<1$; $b(t)\geq0$; $P(t)\geq0$; $Q(t)\geq0$; $A(t)>0$, $A^{\Delta}(t)\geq0$ and $-b/A\in\Re^+$.
- (H_4) There exist constants $L>0$ and $\eta>0$ such that $F(u)/u\geq L$ ($u\neq0$), $f(u)/u\leq\eta$ ($u\neq0$) and $\Phi(t)=LP(t)-\eta Q(t)>0$.

By a solution of (1.1), we mean a nontrivial real-valued function $x(t)$ satisfying (1.1) for $t\in\mathbf{T}$. We recall that a solution $x(t)$ of (1.1) is said to be oscillatory on $[t_0,+\infty)_{\mathbf{T}}$ if it is neither eventually positive nor eventually negative; otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory. Our attention is restricted to those solutions $x(t)$ of (1.1) where $x(t)$ is not eventually identically zero.

Note that if $\mathbf{T}=\mathbf{R}$, then $\sigma(t)=t$, $\mu(t)=0$, $x^{\Delta}(t)=x'(t)$, and (1.1) becomes second-order differential equation

$$\begin{aligned}&\bigl[A(t)\varphi\bigl(y'(t)\bigr)\bigr]'+b(t)\varphi\bigl(y'(t)\bigr)+P(t)F\bigl(\varphi\bigl(x\bigl(\delta(t)\bigr)\bigr)\bigr)-Q(t)f\bigl(\varphi\bigl(x\bigl(\gamma(t)\bigr)\bigr)\bigr)=0,\\&\quad t\in\mathbf{R}.\end{aligned}\tag{1.2}$$

If $\mathbf{T}=\mathbf{Z}$, then $\sigma(t)=t+1$, $\mu(t)=1$, $x^{\Delta}(t)=\Delta x(t)=x(t+1)-x(t)$, and (1.1) becomes second-order difference equation

$$\begin{aligned}&\Delta\bigl[A(t)\varphi\bigl(\Delta y(t)\bigr)\bigr]+b(t)\varphi\bigl(\Delta y(t)\bigr)+P(t)F\bigl(\varphi\bigl(x\bigl(\delta(t)\bigr)\bigr)\bigr)-Q(t)f\bigl(\varphi\bigl(x\bigl(\gamma(t)\bigr)\bigr)\bigr)=0,\\&\quad t\in\mathbf{Z}.\end{aligned}\tag{1.3}$$

If $\mathbf{T}=q^{\mathbf{N}_0}=\{q^t:t\in\mathbf{N}_0,q>1\}$, then $\sigma(t)=qt$, $\mu(t)=(q-1)t$, $x^{\Delta}(t)=\Delta_q x(t)=\frac{x(qt)-x(t)}{(q-1)t}$, and (1.1) becomes a second-order q-difference equation

$$\Delta_q\bigl[A(t)\varphi\bigl(\Delta_q y(t)\bigr)\bigr]+b(t)\varphi\bigl(\Delta_q y(t)\bigr)+P(t)F\bigl(\varphi\bigl(x\bigl(\delta(t)\bigr)\bigr)\bigr)-Q(t)f\bigl(\varphi\bigl(x\bigl(\gamma(t)\bigr)\bigr)\bigr)=0.\tag{1.4}$$

In (1.1), if $\lambda = 1$, $Q(t) \equiv 0$, then (1.1) is simplified to the equation

$$\begin{aligned} &\bigl\{A(t)\bigl[x(t)+B(t)x\bigl(\tau(t)\bigr)\bigr]^{\Delta}\bigr\}^{\Delta}+b(t)\bigl[x(t)+B(t)x\bigl(\tau(t)\bigr)\bigr]+P(t)F\bigl(x\bigl(\delta(t)\bigr)\bigr)=0, \\ &t\in\mathbf{T}, t\geq t_0. \end{aligned} \tag{1.5}$$

If $B(t) \equiv 0$ and $Q(t) \equiv 0$ in (1.1), then (1.1) is simplified to the equation

$$\begin{aligned} &\bigl[A(t)\bigl|x^{\Delta}(t)\bigr|^{\lambda-1}x^{\Delta}(t)\bigr]^{\Delta}+b(t)\bigl|x^{\Delta}(t)\bigr|^{\lambda-1}x^{\Delta}(t)+P(t)\bigl|x\bigl(\delta(t)\bigr)\bigr|^{\lambda-1}x\bigl(\delta(t)\bigr)=0, \\ &t\in\mathbf{T}, t\geq t_0. \end{aligned} \tag{1.6}$$

In (1.5), if $b(t) \equiv 0$ and $B(t) \equiv 0$, then (1.5) is simplified to the equation

$$\bigl[A(t)x^{\Delta}(t)\bigr]^{\Delta}+P(t)F\bigl(x\bigl(\delta(t)\bigr)\bigr)=0, \quad t\in\mathbf{T}, t\geq t_0. \tag{1.7}$$

In (1.1), if $B(t) \equiv 0$, $b(t) \equiv 0$, $F(u) = u$, $\delta(t) = t$, $Q(t) \equiv 0$, and λ is an odd, then (1.1) is simplified to the equation

$$\bigl\{A(t)\bigl[x^{\Delta}(t)\bigr]^{\lambda-1}\bigr\}^{\Delta}+P(t)\bigl[x(t)\bigr]^{\lambda-1}=0, \quad t\in\mathbf{T}, t\geq t_0. \tag{1.8}$$

In (1.5), if $b(t) \equiv 0$, $B(t) \equiv 0$ and $A(t) \equiv 1$, then (1.5) is simplified to the equation

$$x^{\Delta\Delta}(t)+P(t)F\bigl(x\bigl(\delta(t)\bigr)\bigr)=0, \quad t\in\mathbf{T}, t\geq t_0. \tag{1.9}$$

In recent years, there has been an increasing interest in studying the oscillatory behavior of first and second-order neutral delay dynamic equations on time scales, see [1–26]. As a special case of (1.1), Agarwal *et al.* [12], Sahiner [13] considered second-order delay dynamic equation (1.9), and Saker [14] considered second-order half-linear dynamic equation (1.8), and established some sufficient conditions for oscillation of (1.8) and (1.9). Erbe *et al.* [16] considered the general nonlinear delay dynamic equation (1.7), setting out to obtain some new oscillation criteria which improve the results given in [13]. So far, oscillation of the second-order nonlinear delay dynamic equation with damping (1.5) has rarely been discussed. Sun *et al.* [10] extended and improved the results of [12–16], meanwhile obtaining some oscillatory criteria. On this basis, by using the Riccati transformation technique and inequalities, Zhang *et al.* [21, 22, 25] studied the oscillatory behavior of all solutions of (1.6). Note that the results in [21, 22, 25] are based on the condition $\delta(\mathbf{T}) = \mathbf{T}$, which can be a restrictive condition and it is not easy to satisfy. For instance, when $\mathbf{T} = \{1, 3, 5, 7, \ldots\}$ and letting $\delta(t) = t - 1$, then δ is a strictly increasing function, $\delta(t) \leq t$ and $\lim_{t\to+\infty} \delta(t) = +\infty$, but $\delta(\mathbf{T}) = \{0, 2, 4, 6, \ldots\} \neq \mathbf{T}$, so the condition $\delta(\mathbf{T}) = \mathbf{T}$ in [21, 22, 25] does not hold and the results in [21, 22, 25] may not be true.

Hence, it would be interesting to study the oscillation behavior of (1.1) when $\delta(\mathbf{T}) = \mathbf{T}$ does not hold. In this paper, we discuss the oscillation of solutions of (1.1). By using the generalized Riccati transformation and the inequality technique, we obtain some new oscillation criteria for (1.1), some results of [10–17, 21, 22, 25, 28] are now special examples of our results. We shall also consider the two cases

$$\int_{t_0}^{+\infty}\left[\frac{e_{-b/A}(s,t_0)}{A(s)}\right]^{1/\lambda}\Delta s=+\infty \tag{1.10}$$

and

$$\int_{t_0}^{+\infty}\left[\frac{e_{-b/A}(s,t_0)}{A(s)}\right]^{1/\lambda}\Delta s<+\infty. \tag{1.11}$$

2 Preliminaries

We shall employ the following lemmas.

Lemma 2.1 [4] *Assume that $x(t)$ is Δ-differentiable and eventually positive or eventually negative, then*

$$\left[x^{\lambda}(t)\right]^{\Delta}=\lambda\int_0^1\left[hx^{\sigma}+(1-h)x\right]^{\lambda-1}x^{\Delta}(t)\,dh. \tag{2.1}$$

Lemma 2.2 [11] *Assume that*

(i) $u\in C_{rd}^2(\mathbf{I},\mathbf{R})$, *where* $\mathbf{I}=[t^*,+\infty)$, $t^*>0$.
(ii) $u(t)>0$, $u^{\Delta}(t)>0$, $u^{\Delta\Delta}(t)\le 0$, $t\ge t^*$.

Then, for every $c\in(0,1)$, there exists a constant $t_c\in\mathbf{T}$, $t_c>t^*$ such that $u(\sigma(t))\le\frac{\sigma(t)u(\delta(t))}{c\delta(t)}$ for all $t\ge t_c$.

Lemma 2.3 [4] *If $g\in\Re^+$, i.e., $g:\mathbf{T}\to\mathbf{R}$ is rd-continuous and such that $1+\mu(t)g(t)>0$ for all $t\in[t_0,+\infty)_{\mathbf{T}}$, then the initial value problem $y^{\Delta}(t)=g(t)y(t)$, $y(t_0)=y_0\in\mathbf{R}$ has a unique and positive solution on $[t_0,+\infty)_{\mathbf{T}}$, denoted by $e_g(t,t_0)$. This 'exponential function' satisfies the semigroup property $e_g(a,b)e_g(b,c)=e_g(a,c)$.*

Lemma 2.4 [18] *Assume that a and b are nonnegative real numbers, then $rab^{r-1}-a^r\le(r-1)b^r$ for all $r>1$, where the equality holds if and only if $a=b$.*

Lemma 2.5 ([19], Hölder's inequality) *Let $a,b\in\mathbf{T}$ and $a<b$. For rd-continuous functions $f,g:[a,b]\to\mathbf{R}$, we have $\int_a^b|f(u)g(u)|\Delta u\le(\int_a^b|f(u)|^p\Delta u)^{1/p}(\int_a^b|g(u)|^q\Delta u)^{1/q}$, where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.*

Lemma 2.6 *Assume* (H_1)-(H_4) *and* (1.10). *Let $x(t)$ be an eventually positive solution of* (1.1). *Then there exists $t_1\in[t_0,+\infty)_{\mathbf{T}}$ such that*

$$\begin{aligned}&y(t)>0,\qquad y^{\Delta}(t)>0,\qquad A(t)\varphi\bigl(y^{\Delta}(t)\bigr)>0,\\&\bigl[A(t)\varphi\bigl(y^{\Delta}(t)\bigr)\bigr]^{\Delta}\le 0\quad\textit{and}\quad x(t)\ge\bigl[1-B(t)\bigr]y(t)\end{aligned}$$

for all $t\in[t_1,+\infty)_{\mathbf{T}}$.

Proof Since $x(t)$ is an eventually positive solution of (1.1), there exists $t_1\in[t_0,+\infty)_{\mathbf{T}}$ such that $x(t)>0$, $x(\tau(t))>0$, $x(\gamma(t))=x(\delta(t))>0$ for all $t\in[t_1,+\infty)_{\mathbf{T}}$, thus, $y(t)>0$. From (1.1) we obtain

$$\begin{aligned}&\bigl[A(t)\varphi\bigl(y^{\Delta}(t)\bigr)\bigr]^{\Delta}+b(t)\varphi\bigl(y^{\Delta}(t)\bigr)\\&\quad\le -LP(t)\bigl[x\bigl(\delta(t)\bigr)\bigr]^{\lambda}+\eta Q(t)\bigl[x\bigl(\gamma(t)\bigr)\bigr]^{\lambda}\\&\quad=-\bigl[LP(t)-\eta Q(t)\bigr]\bigl[x\bigl(\delta(t)\bigr)\bigr]^{\lambda}=-\Phi(t)\bigl[x\bigl(\delta(t)\bigr)\bigr]^{\lambda}<0,\quad t\in[t_1,+\infty)_{\mathbf{T}}.\end{aligned}\tag{2.2}$$

Thus, by Lemma 2.3, we obtain on $[t_1, +\infty)_{\mathbf{T}}$,

$$\begin{aligned}\left[\frac{A(t)\varphi(y^\Delta(t))}{e_{-b/A}(t,t_0)}\right]^\Delta &= \frac{[A(t)\varphi(y^\Delta(t))]^\Delta e_{-b/A}(t,t_0) - A(t)\varphi(y^\Delta(t))[e_{-b/A}(t,t_0)]^\Delta}{e_{-b/A}(t,t_0)e_{-b/A}(\sigma(t),t_0)} \\ &= \frac{[A(t)\varphi(y^\Delta(t))]^\Delta + b(t)\varphi(y^\Delta(t))}{e_{-b/A}(\sigma(t),t_0)} \le -\frac{\Phi(t)[x(\delta(t))]^\lambda}{e_{-b/A}(\sigma(t),t_0)} < 0. \qquad (2.3)\end{aligned}$$

Hence, $\frac{A(t)\varphi(y^\Delta(t))}{e_{-b/A}(t,t_0)} = \frac{A(t)|y^\Delta(t)|^{\lambda-1}y^\Delta(t)}{e_{-b/A}(t,t_0)}$ is decreasing and, therefore, eventually of one sign, so (use Lemma 2.3) $y^\Delta(t)$ is either eventually positive or eventually negative. We assert that $y^\Delta(t) > 0$ for all $t \in [t_1, +\infty)_{\mathbf{T}}$. Assume that $y^\Delta(t) < 0$ eventually; then there exists $t_2 \in [t_1, +\infty)_{\mathbf{T}}$ such that $y^\Delta(t) < 0$ for all $t \in [t_2, +\infty)_{\mathbf{T}}$. From Lemma 2.3 we obtain

$$\frac{A(t)\varphi(y^\Delta(t))}{e_{-b/A}(t,t_0)} \le \frac{A(t_2)\varphi(y^\Delta(t_2))}{e_{-b/A}(t_2,t_0)} = -M < 0 \quad \text{for all } t \in [t_2, +\infty)_{\mathbf{T}},$$

where $M = -\frac{A(t_2)\varphi(y^\Delta(t_2))}{e_{-b/A}(t_2,t_0)} = \frac{A(t_2)|y^\Delta(t_2)|^{\lambda-1}[-y^\Delta(t_2)]}{e_{-b/A}(t_2,t_0)} > 0$. By the above inequality we obtain $A(t)\varphi(y^\Delta(t)) \le -Me_{-b/A}(t,t_0)$, therefore, it follows that $[-y^\Delta(t)]^\lambda \ge \frac{Me_{-b/A}(t,t_0)}{A(t)}$, *i.e.*,

$$y^\Delta(t) \le -M^{\frac{1}{\lambda}}\left[\frac{e_{-b/A}(t,t_0)}{A(t)}\right]^{1/\lambda} \quad \text{for all } t \in [t_2, +\infty)_{\mathbf{T}}.$$

Integrating the above inequality from t_2 to t ($t \in [t_2, +\infty)_{\mathbf{T}}$) leads to

$$y(t) \le y(t_2) - M^{\frac{1}{\lambda}}\int_{t_2}^t \left[\frac{e_{-b/A}(s,t_0)}{A(s)}\right]^{1/\lambda}\Delta s \to -\infty \quad \text{as } t \to +\infty,$$

which contradicts with $y(t) > 0$. So $y^\Delta(t) > 0$, and then $A(t)\varphi(y^\Delta(t)) > 0$. From (2.2) we obtain $[A(t)\varphi(y^\Delta(t))]^\Delta \le 0$.

Moreover, in view of $x(t) \le y(t)$, we have

$$y(t) = x(t) + B(t)x(\tau(t)) \le x(t) + B(t)y(\tau(t)) \le x(t) + B(t)y(t),$$

i.e., $x(t) \ge [1 - B(t)]y(t)$. This completes the proof. □

3 Main results

In this section, we use the generalized Riccati transformation and the inequality technique to obtain some sufficient conditions for oscillation of all solutions of (1.1), provided the condition (1.10) holds. If (1.10) is not satisfied, *i.e.*, if (1.11) holds, then we present some conditions that guarantee that each solution of (1.1) is either oscillatory or converges to zero.

For convenience, consider the set $D = \{(t,s) : t \ge s \ge t_0, t, s \in [t_0, +\infty)_{\mathbf{T}}\}$. We say that a function $H = H(t,s)$ belongs to function class Ω, denoted by $H \in \Omega$, if $H \in C_{rd}(D, \mathbf{R})$, which satisfies

$$H(t,t) = 0 \quad \text{for } t \ge t_0, \qquad H(t,s) > 0 \quad \text{for } t > s \ge t_0, t, s \in [t_0, +\infty)_{\mathbf{T}}$$

and has a nonpositive continuous Δ-partial derivative $H_s^\Delta(t,s)$ with respect to the second variable, *i.e.*, $H_s^\Delta(t,s) \in C_{rd}$ and $H_s^\Delta(t,s) \le 0$.

Theorem 3.1 *Assume* (H_1)-(H_4) *and* (1.10). *If there exist* $H \in \Omega$ *and a positive and* Δ-*differentiable function* $\phi : \mathbf{T} \to \mathbf{R}$ *such that*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{T_0}^{t} \left\{ c^{\lambda} H(t,s)\phi(s)\Psi(s) - \frac{[|h(t,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\phi(s)A(s)H(t,s)]^{\lambda}} \right\} \Delta s = +\infty \tag{3.1}$$

for some constant $T_0 \geq t_0$, *where the constant c is defined as in Lemma* 2.2, *the functions* $\Psi(s)$ *and* $h(t,s)$ *are defined as follows*:

$$\begin{aligned} \Psi(s) &= \Phi(s)\big[1 - B\big(\delta(s)\big)\big]^{\lambda} \left[\frac{\delta(s)}{\sigma(s)}\right]^{\lambda}, \\ h(t,s) &= \big[H(t,s)\big]_s^{\Delta} + \left[\phi^{\Delta}(s) - \frac{\phi(s)b(s)}{A(\sigma(s))}\right] \frac{H(t,s)}{\phi(\sigma(s))}. \end{aligned} \tag{3.2}$$

Then (1.1) *is oscillatory on* $[t_0, +\infty)_{\mathbf{T}}$.

Proof Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, +\infty)_{\mathbf{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ for all $t \in [t_1, +\infty)_{\mathbf{T}}$, $t_1 \in [t_0, +\infty)_{\mathbf{T}}$. Then $y(t) > 0$. Now, we claim that

$$y^{\Delta\Delta}(t) \leq 0 \quad \text{and} \quad \begin{cases} [(y(t))^{\lambda}]^{\Delta} \geq \lambda (y(t))^{\lambda-1} y^{\Delta}(t), & \text{if } \lambda \geq 1, \\ [(y(t))^{\lambda}]^{\Delta} \geq \lambda [y(\sigma(t))]^{\lambda-1} y^{\Delta}(t), & \text{if } 0 < \lambda < 1. \end{cases} \tag{3.3}$$

In fact, if $\lambda \geq 1$, then, by (2.1), we get

$$\big[\big(y(t)\big)^{\lambda}\big]^{\Delta} \geq \lambda \int_0^1 \big[hy + (1-h)y\big]^{\lambda-1} y^{\Delta}(t)\, dh = \lambda \big(y(t)\big)^{\lambda-1} y^{\Delta}(t),$$

and then $[(y^{\Delta}(t))^{\lambda}]^{\Delta} \geq \lambda (y^{\Delta}(t))^{\lambda-1} y^{\Delta\Delta}(t)$. Hence,

$$\begin{aligned} \big[A(t)\big(y^{\Delta}(t)\big)^{\lambda}\big]^{\Delta} &= A^{\Delta}(t)\big(y^{\Delta}(t)\big)^{\lambda} + A\big(\sigma(t)\big)\big[\big(y^{\Delta}(t)\big)^{\lambda}\big]^{\Delta} \\ &\geq A^{\Delta}(t)\big(y^{\Delta}(t)\big)^{\lambda} + \lambda A\big(\sigma(t)\big)\big(y^{\Delta}(t)\big)^{\lambda-1} y^{\Delta\Delta}(t), \end{aligned}$$

in view of Lemma 2.6 and $A^{\Delta}(t) \geq 0$, we get $y^{\Delta\Delta}(t) \leq 0$.

If $0 < \lambda < 1$, then, by (2.1), we get

$$\big[\big(y(t)\big)^{\lambda}\big]^{\Delta} \geq \lambda \int_0^1 \big[hy^{\sigma} + (1-h)y^{\sigma}\big]^{\lambda-1} y^{\Delta}(t)\, dh = \lambda \big[y\big(\sigma(t)\big)\big]^{\lambda-1} y^{\Delta}(t).$$

Similarly, we can easily find $y^{\Delta\Delta}(t) \leq 0$, and hence this completes the proof of the claim.

By Lemma 2.2, there exists $T_0 \in [t_1, +\infty)_{\mathbf{T}}$ with $T_0 \geq \max\{t_c, t_1\}$, $\forall c \in (0,1)$, we have

$$y\big(\delta(t)\big) \geq \frac{c\delta(t)}{\sigma(t)} y\big(\sigma(t)\big) \geq \frac{c\delta(t)}{\sigma(t)} y(t) \quad \text{for all } t \in [T_0, +\infty)_{\mathbf{T}}. \tag{3.4}$$

Now define the function $w(t)$ by

$$w(t) = \phi(t) \frac{A(t)\varphi(y^{\Delta}(t))}{\varphi(y(t))} = \phi(t) \frac{A(t)(y^{\Delta}(t))^{\lambda}}{(y(t))^{\lambda}} \quad \text{on } [T_0, +\infty)_{\mathbf{T}}. \tag{3.5}$$

Then we have $w(t) > 0$ $(t \in [T_0, +\infty)_{\mathbf{T}})$.

If $\lambda \geq 1$, then, by (3.5) (use (2.2), Lemma 2.6 and $[(y(t))^{\lambda}]^{\Delta} \geq \lambda(y(t))^{\lambda-1}y^{\Delta}(t)$), we obtain for $t \in [T_0, +\infty)_{\mathbf{T}}$

$$\begin{aligned}
w^{\Delta}(t) &= \phi^{\Delta}(t)\frac{A(\sigma(t))\varphi(y^{\Delta}(\sigma(t)))}{\varphi(y(\sigma(t)))} \\
&\quad + \phi(t)\frac{[A(t)(y^{\Delta}(t))^{\lambda}]^{\Delta}(y(t))^{\lambda} - A(t)(y^{\Delta}(t))^{\lambda}[(y(t))^{\lambda}]^{\Delta}}{(y(t))^{\lambda}[y(\sigma(t))]^{\lambda}} \\
&\leq \frac{\phi^{\Delta}(t)w(\sigma(t))}{\phi(\sigma(t))} - \phi(t)\frac{\Phi(t)[x(\delta(t))]^{\lambda} + b(t)\varphi(y^{\Delta}(t))}{[y(\sigma(t))]^{\lambda}} - \phi(t)\frac{A(t)(y^{\Delta}(t))^{\lambda}[(y(t))^{\lambda}]^{\Delta}}{(y(t))^{\lambda}[y(\sigma(t))]^{\lambda}} \\
&\leq \frac{\phi^{\Delta}(t)w(\sigma(t))}{\phi(\sigma(t))} - \phi(t)\Phi(t)\bigl[1 - B\bigl(\delta(t)\bigr)\bigr]^{\lambda}\frac{[y(\delta(t))]^{\lambda}}{[y(\sigma(t))]^{\lambda}} \\
&\quad - \phi(t)\frac{b(t)[y^{\Delta}(t)]^{\lambda}}{[y(\sigma(t))]^{\lambda}} - \lambda\phi(t)\frac{A(t)(y^{\Delta}(t))^{\lambda+1}}{y(t)[y(\sigma(t))]^{\lambda}}.
\end{aligned} \tag{3.6}$$

From $y^{\Delta\Delta}(t) \leq 0$ and $y^{\Delta}(t) > 0$, we get $y^{\Delta}(t) \geq y^{\Delta}(\sigma(t))$ and $y(t) \leq y(\sigma(t))$. In view of the first formula of (3.2) and (3.4), it follows from (3.6) that

$$\begin{aligned}
w^{\Delta}(t) &\leq \frac{\phi^{\Delta}(t)w(\sigma(t))}{\phi(\sigma(t))} - \phi(t)\Phi(t)\bigl[1 - B\bigl(\delta(t)\bigr)\bigr]^{\lambda}\left[\frac{c\delta(t)}{\sigma(t)}\right]^{\lambda} \\
&\quad - \phi(t)\frac{b(t)[y^{\Delta}(\sigma(t)]^{\lambda}}{[y(\sigma(t))]^{\lambda}} - \lambda\phi(t)\frac{A(t)[y^{\Delta}(\sigma(t))]^{\lambda+1}}{[y(\sigma(t))]^{\lambda+1}} \\
&= \left[\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))} - \frac{\phi(t)b(t)}{\phi(\sigma(t))A(\sigma(t))}\right]w\bigl(\sigma(t)\bigr) - c^{\lambda}\phi(t)\Psi(t) \\
&\quad - \frac{\lambda\phi(t)A(t)[w(\sigma(t))]^{\frac{\lambda+1}{\lambda}}}{[\phi(\sigma(t))A(\sigma(t))]^{\frac{\lambda+1}{\lambda}}} \quad \text{for } t \in [T_0, +\infty)_{\mathbf{T}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
c^{\lambda}\phi(t)\Psi(t) &\leq -w^{\Delta}(t) + \left[\phi^{\Delta}(t) - \frac{\phi(t)b(t)}{A(\sigma(t))}\right]\frac{w(\sigma(t))}{\phi(\sigma(t))} \\
&\qquad - \lambda\phi(t)A(t)\left[\frac{w(\sigma(t))}{\phi(\sigma(t))A(\sigma(t))}\right]^{\frac{\lambda+1}{\lambda}} \quad \text{for } t \in [T_0, +\infty)_{\mathbf{T}}.
\end{aligned} \tag{3.7}$$

If $0 < \lambda < 1$, in view of $[(y(t))^{\lambda}]^{\Delta} \geq \lambda[y(\sigma(t))]^{\lambda-1}y^{\Delta}(t)$, then (3.6) becomes

$$\begin{aligned}
w^{\Delta}(t) &\leq \frac{\phi^{\Delta}(t)w(\sigma(t))}{\phi(\sigma(t))} - \phi(t)\Phi(t)\bigl[1 - B\bigl(\delta(t)\bigr)\bigr]^{\lambda}\frac{[y(\delta(t))]^{\lambda}}{[y(\sigma(t))]^{\lambda}} \\
&\quad - \phi(t)\frac{b(t)[y^{\Delta}(t)]^{\lambda}}{[y(\sigma(t))]^{\lambda}} - \lambda\phi(t)\frac{A(t)(y^{\Delta}(t))^{\lambda+1}}{(y(t))^{\lambda}y(\sigma(t))},
\end{aligned}$$

similarly, we can get (3.7). Then from (3.7), we can obtain

$$\begin{aligned}
&\int_{T_0}^{t} c^{\lambda}H(t,s)\phi(s)\Psi(s)\Delta s \\
&\quad \leq -\int_{T_0}^{t} H(t,s)w^{\Delta}(s)\Delta s + \int_{T_0}^{t} H(t,s)\left[\phi^{\Delta}(s) - \frac{\phi(s)b(s)}{A(\sigma(s))}\right]\frac{w(\sigma(s))}{\phi(\sigma(s))}\Delta s
\end{aligned}$$

$$
\begin{aligned}
&-\int_{T_0}^{t} H(t,s)\frac{\lambda\phi(s)A(s)}{[\phi(\sigma(s))A(\sigma(s))]^{\frac{\lambda+1}{\lambda}}}\big[w\big(\sigma(s)\big)\big]^{\frac{\lambda+1}{\lambda}}\Delta s \\
&= -\big[H(t,s)w(s)\big]_{T_0}^{t} + \int_{T_0}^{t}\big[H(t,s)\big]_s^{\Delta} w\big(\sigma(s)\big)\Delta s \\
&\quad + \int_{T_0}^{t} H(t,s)\bigg[\phi^{\Delta}(s) - \frac{\phi(s)b(s)}{A(\sigma(s))}\bigg]\frac{w(\sigma(s))}{\phi(\sigma(s))}\Delta s \\
&\quad - \int_{T_0}^{t} H(t,s)\frac{\lambda\phi(s)A(s)}{[\phi(\sigma(s))A(\sigma(s))]^{\frac{\lambda+1}{\lambda}}}\big[w\big(\sigma(s)\big)\big]^{\frac{\lambda+1}{\lambda}}\Delta s \\
&\le H(t,T_0)w(T_0) + \int_{T_0}^{t}\bigg\{\big[H(t,s)\big]_s^{\Delta} + \bigg[\phi^{\Delta}(s) - \frac{\phi(s)b(s)}{A(\sigma(s))}\bigg]\frac{H(t,s)}{\phi(\sigma(s))}\bigg\} w\big(\sigma(s)\big)\Delta s \\
&\quad - \int_{T_0}^{t} H(t,s)\frac{\lambda\phi(s)A(s)}{[\phi(\sigma(s))A(\sigma(s))]^{\frac{\lambda+1}{\lambda}}}\big[w\big(\sigma(s)\big)\big]^{\frac{\lambda+1}{\lambda}}\Delta s \\
&= H(t,T_0)w(T_0) + \int_{T_0}^{t} h(t,s)w\big(\sigma(s)\big)\Delta s \\
&\quad - \int_{T_0}^{t}\frac{\lambda\phi(s)A(s)H(t,s)}{[\phi(\sigma(s))A(\sigma(s))]^{\frac{\lambda+1}{\lambda}}}\big[w\big(\sigma(s)\big)\big]^{\frac{\lambda+1}{\lambda}}\Delta s. \qquad (3.8)
\end{aligned}
$$

Now, in Lemma 2.4, we let

$$
\begin{aligned}
&r = \frac{\lambda+1}{\lambda}, \qquad a = \big[\lambda\phi(s)A(s)H(t,s)\big]^{\frac{\lambda}{\lambda+1}}\frac{w(\sigma(s))}{\phi(\sigma(s))A(\sigma(s))}, \\
&b = \bigg(\frac{\lambda}{\lambda+1}\bigg)^{\lambda}\big[\big|h(t,s)\big|\phi\big(\sigma(s)\big)A\big(\sigma(s)\big)\big]^{\lambda}\big[\lambda\phi(s)A(s)H(t,s)\big]^{\frac{-\lambda^2}{\lambda+1}}.
\end{aligned}
$$

From Lemma 2.4, we then obtain

$$
\big|h(t,s)\big|w\big(\sigma(s)\big) - \frac{\lambda\phi(s)A(s)H(t,s)}{[\phi(\sigma(s))A(\sigma(s))]^{\frac{\lambda+1}{\lambda}}}\big[w\big(\sigma(s)\big)\big]^{\frac{\lambda+1}{\lambda}} \le \frac{[|h(t,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\phi(s)A(s)H(t,s)]^{\lambda}}.
$$

Hence, (3.8) implies

$$
\int_{T_0}^{t} c^{\lambda}H(t,s)\phi(s)\Psi(s)\Delta s \le H(t,T_0)w(T_0) + \int_{T_0}^{t}\frac{[|h(t,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\phi(s)A(s)H(t,s)]^{\lambda}}\Delta s,
$$

and therefore

$$
\begin{aligned}
&\int_{T_0}^{t}\bigg\{c^{\lambda}H(t,s)\phi(s)\Psi(s) - \frac{[|h(t,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\phi(s)A(s)H(t,s)]^{\lambda}}\bigg\}\Delta s \\
&\quad \le H(t,T_0)w(T_0) \le H(t,t_0)w(T_0). \qquad (3.9)
\end{aligned}
$$

Thus

$$
\frac{1}{H(t,t_0)}\int_{T_0}^{t}\bigg\{c^{\lambda}H(t,s)\phi(s)\Psi(s) - \frac{[|h(t,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\phi(s)A(s)H(t,s)]^{\lambda}}\bigg\}\Delta s \le w(T_0),
$$

contradicting (3.1). This completes the proof. □

Remark 3.1 From Theorem 3.1, we can obtain different conditions for oscillation of all solutions of (1.1) with different choices of $H(t,s)$ and $\phi(t)$. For example, $H(t,s) = (t-s)^{\omega}$, $H(t,s) = (\ln \frac{t+1}{s+1})^{\omega}$, or $H(t,s) = [\int_s^t u(t)\Delta t]^{\omega}$ ($u(t) > 0$), etc. Now, let us consider the function $H(t,s)$ defined by

$$H(t,s) = (t-s)^{\omega}, \quad \omega \geq 1, t \geq s \geq t_0, t,s \in [t_0,+\infty)_{\mathbf{T}}.$$

Then using the same idea as in the proof of Theorem 3.1, we can now obtain the following result.

Theorem 3.2 *Assume* (H_1)-(H_4) *and* (1.10). *If there exists a positive and* Δ*-differentiable function* $\phi : \mathbf{T} \to \mathbf{R}$ *and* $\omega \geq 1$ *such that*

$$\begin{aligned}\limsup_{t\to+\infty} \frac{1}{t^{\omega}} \int_{T_0}^{t} (t-s)^{\omega} \Big\{ & c^{\lambda}\phi(s)\Psi(s) \\ & - \frac{(\lambda+1)^{-(\lambda+1)}}{[\phi(s)A(s)]^{\lambda}} \left[\left| \phi^{\Delta}(s) - \frac{\phi(s)b(s)}{A(\sigma(s))} \right| A\big(\sigma(s)\big) \right]^{\lambda+1} \Big\} \Delta s = +\infty \end{aligned} \tag{3.10}$$

for some constant $T_0 \geq t_0$, *where constant* c *is defined as in Lemma* 2.2, $\Psi(s)$ *is defined as in Theorem* 3.1, *then* (1.1) *is oscillatory on* $[t_0,+\infty)_{\mathbf{T}}$.

If (3.1) does not hold, then we have the following results.

Theorem 3.3 *Assume* (H_1)-(H_4) *and* (1.10). *If there exist functions* $H \in \Omega$, $\zeta_1(t), \zeta_2(t) \in C_{rd}(\mathbf{T},\mathbf{R})$, *and a positive, nondecreasing, differentiable function* $\phi \in C_{rd}(\mathbf{T},\mathbf{R}^{+})$ *such that*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_u^t H(t,s)\phi(s)\Psi(s)\Delta s \geq \zeta_1(u), \tag{3.11}$$

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_u^t \frac{[|h(t,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{[\phi(s)A(s)H(t,s)]^{\lambda}} \Delta s \leq \zeta_2(u) \tag{3.12}$$

for $u \geq T_0$, *and*

$$\liminf_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{T_0}^t \frac{H(t,s)\phi(s)A(s)[\zeta_1(\sigma(s)) - \theta\zeta_2(\sigma(s))]_{+}^{(1+\lambda)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}} \Delta s = +\infty \tag{3.13}$$

for some constant $T_0 \geq t_0$, *where* $[\zeta_1(\sigma(s)) - \theta\zeta_2(\sigma(s))]_{+} = \max\{[\zeta_1(\sigma(s)) - \theta\zeta_2(\sigma(s))], 0\}$, $\theta = c^{-\lambda}(\lambda+1)^{-(\lambda+1)}$, *the constant* c *is defined as in Lemma* 2.2, *the functions* $\Psi(s)$ *and* $h(t,s)$ *are defined as in Theorem* 3.1, *then* (1.1) *is oscillatory on* $[t_0,+\infty)_{\mathbf{T}}$.

Proof Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,+\infty)_{\mathbf{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ for all $t \in [t_1,+\infty)_{\mathbf{T}}$, $t_1 \in [t_0,+\infty)_{\mathbf{T}}$. We proceed as in the proof of Theorem 3.1 to obtain (3.8) and (3.9). Then from (3.9), we have for $t \geq u \geq T_0$

$$\int_u^t c^{\lambda}H(t,s)\phi(s)\Psi(s)\Delta s - \int_u^t \frac{[|h(t,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\phi(s)A(s)H(t,s)]^{\lambda}} \Delta s \leq H(t,t_0)w(u),$$

thus,

$$\begin{aligned}&\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_u^t c^\lambda H(t,s)\phi(s)\Psi(s)\Delta s\\&\quad\le w(u)+\limsup_{t\to+\infty}\int_u^t\frac{[|h(t,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\phi(s)A(s)H(t,s)]^\lambda}\Delta s.\end{aligned}$$

Then, in view of (3.11) and (3.12), from the above inequality, we have $c^\lambda\zeta_1(u)\le w(u)+\frac{\zeta_2(u)}{(\lambda+1)^{\lambda+1}}$, that is,

$$\zeta_1(u)-\theta\zeta_2(u)\le c^{-\lambda}w(u),\quad u\ge T_0\ge t_0. \tag{3.14}$$

Moreover, from (3.8), we find that

$$\begin{aligned}&\frac{1}{H(t,t_0)}\int_{T_0}^t\left\{\frac{\lambda\phi(s)A(s)H(t,s)[w(\sigma(s))]^{(\lambda+1)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}-\left|h(t,s)\right|w\bigl(\sigma(s)\bigr)\right\}\Delta s\\&\quad\le w(T_0)-\frac{1}{H(t,t_0)}\int_{T_0}^t c^\lambda H(t,s)\phi(s)\Psi(s)\Delta s,\end{aligned}$$

this inequality implies

$$\begin{aligned}&\liminf_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{T_0}^t\left\{\frac{\lambda\phi(s)A(s)H(t,s)[w(\sigma(s))]^{(\lambda+1)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}-\left|h(t,s)\right|w\bigl(\sigma(s)\bigr)\right\}\Delta s\\&\quad\le C_0,\end{aligned}\tag{3.15}$$

where $C_0=w(T_0)-c^\lambda\zeta_1(T_0)$ is a constant. Now, we claim that

$$\liminf_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{T_0}^t\frac{\lambda\phi(s)A(s)H(t,s)[w(\sigma(s))]^{(\lambda+1)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s<+\infty. \tag{3.16}$$

In fact, if (3.16) does not hold, then there exists a sequence $\{T_n\}_{n=1}^{+\infty}:T_n\in[t_2,+\infty)_{\mathbf{T}}$ with $\lim_{n\to+\infty}T_n=+\infty$ such that $\lim_{n\to+\infty}\frac{1}{H(T_n,t_0)}\int_{T_0}^{T_n}\frac{\lambda\phi(s)A(s)H(T_n,s)[w(\sigma(s))]^{(\lambda+1)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s=+\infty$, it follows from (3.15) that

$$\lim_{n\to+\infty}\frac{1}{H(T_n,t_0)}\int_{T_0}^{T_n}\left|h(T_n,s)\right|w\bigl(\sigma(s)\bigr)\Delta s=+\infty. \tag{3.17}$$

Thus, for all sufficiently large positive integers n,

$$\begin{aligned}&\frac{1}{H(T_n,t_0)}\int_{T_0}^{T_n}\frac{\lambda\phi(s)A(s)H(T_n,s)[w(\sigma(s))]^{(\lambda+1)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s\\&\quad-\frac{1}{H(T_n,t_0)}\int_{T_0}^{T_n}\left|h(T_n,s)\right|w\bigl(\sigma(s)\bigr)\Delta s<C_0+1.\end{aligned}$$

Therefore, for all sufficiently large positive integers n and $\varepsilon\in(0,1)$, we can easily find

$$\frac{\int_{T_0}^{T_n}|h(T_n,s)|w(\sigma(s))\Delta s}{\int_{T_0}^{T_n}\frac{\lambda\phi(s)A(s)H(T_n,s)[w(\sigma(s))]^{(\lambda+1)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s}>1-\varepsilon>0. \tag{3.18}$$

On the other hand, by Lemma 2.5, we obtain

$$\begin{aligned}
&\int_{T_0}^{T_n} \left|h(T_n,s)\right| w\bigl(\sigma(s)\bigr)\Delta s \\
&= \lambda^{\frac{-\lambda}{\lambda+1}} \int_{T_0}^{T_n} \left[\frac{\lambda\phi(s)A(s)H(T_n,s)}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\right]^{\frac{\lambda}{\lambda+1}} w\bigl(\sigma(s)\bigr)\frac{|h(T_n,s)|\phi(\sigma(s))A(\sigma(s))}{[\phi(s)A(s)H(T_n,s)]^{\lambda/(\lambda+1)}}\Delta s \\
&\le \lambda^{\frac{-\lambda}{\lambda+1}} \left\{\int_{T_0}^{T_n} \frac{\lambda\phi(s)A(s)H(T_n,s)[w(\sigma(s))]^{(\lambda+1)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s\right\}^{\frac{\lambda}{\lambda+1}} \\
&\quad\times \left\{\int_{T_0}^{T_n} \frac{[|h(T_n,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{[\phi(s)A(s)H(T_n,s)]^{\lambda}}\Delta s\right\}^{\frac{1}{\lambda+1}}.
\end{aligned}$$

Further, in view of (3.18), one can easily find

$$\begin{aligned}
0 &< \frac{(1-\varepsilon)^{\lambda}}{H(T_n,t_0)} \int_{T_0}^{T_n} \left|h(T_n,s)\right| w\bigl(\sigma(s)\bigr)\Delta s \\
&< \frac{\{\int_{T_0}^{T_n} |h(T_n,s)|w(\sigma(s))\Delta s\}^{\lambda+1}}{H(T_n,t_0)\{\int_{T_0}^{T_n} \frac{\lambda\phi(s)A(s)H(T_n,s)[w(\sigma(s))]^{(\lambda+1)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s\}^{\lambda}} \\
&\le \frac{\lambda^{-\lambda}}{H(T_n,t_0)} \int_{T_0}^{T_n} \frac{[|h(T_n,s)|\phi(\sigma(s))A(\sigma(s))]^{\lambda+1}}{[\phi(s)A(s)H(T_n,s)]^{\lambda}}\Delta s \le \lambda^{-\lambda}\zeta_2(T_0),
\end{aligned}$$

contradicting (3.17). Therefore (3.16) holds. Now, in view of (3.14) and (3.16), it follows that

$$\begin{aligned}
&\liminf_{n\to+\infty} \frac{1}{H(T_n,t_0)} \int_{T_0}^{T_n} \frac{H(T_n,s)\phi(s)A(s)[\zeta_1(\sigma(s))-\theta\zeta_2(\sigma(s))]_+^{(1+\lambda)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s \\
&\quad\le \liminf_{n\to+\infty} \frac{c^{-(\lambda+1)}}{H(T_n,t_0)} \int_{T_0}^{T_n} \frac{H(T_n,s)\phi(s)A(s)[w(\sigma(s))]^{(1+\lambda)/\lambda}}{[\phi(\sigma(s))A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s < +\infty.
\end{aligned}$$

But this contradicts condition (3.13). The proof is complete. □

Clearly, the following immediate result can be extracted from Theorem 3.3.

Theorem 3.4 *Assume* (H$_1$)-(H$_4$) *and* (1.10). *If there exist functions* $\zeta_1(t),\zeta_2(t)\in C_{rd}(\mathbf{T},\mathbf{R})$ *and a constant* $\omega\ge 1$ *such that*

$$\limsup_{t\to+\infty} \frac{1}{t^{\omega}} \int_u^t (t-s)^{\omega}\Psi(s)\Delta s \ge \zeta_1(u), \tag{3.19}$$

$$\limsup_{t\to+\infty} \frac{1}{t^{\omega}} \int_u^t \frac{(t-s)^{\omega}}{A^{\lambda}(s)}\left[\frac{\omega A(\sigma(s))}{t-s}+b(s)\right]^{\lambda+1}\Delta s \le \zeta_2(u) \tag{3.20}$$

for $u\ge T_0$, *and*

$$\liminf_{t\to+\infty} \frac{1}{t^{\omega}} \int_{T_0}^t \frac{(t-s)^{\omega}A(s)[\xi_1(\sigma(s))-\theta\xi_2(\sigma(s))]_+^{(\lambda+1)/\lambda}}{[A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s = +\infty \tag{3.21}$$

for some constant $T_0 \geq t_0$, *where* $[\zeta_1(\sigma(s)) - \theta\zeta_2(\sigma(s))]_+ = \max\{[\zeta_1(\sigma(s)) - \theta\zeta_2(\sigma(s))], 0\}$, $\theta = c^{-\lambda}(\lambda+1)^{-(\lambda+1)}$, *the constant* c *is defined as in Lemma* 2.2, *the function* $\Psi(s)$ *is defined as in Theorem* 3.1, *then* (1.1) *is oscillatory on* $[t_0, +\infty)_{\mathbf{T}}$.

Next, when (1.11) holds, we give some conditions that guarantee that every solution of (1.1) oscillates or converges to zero.

Theorem 3.5 *Assume* (H_1)-(H_4), (1.11); *if there exist* $H \in \Omega$ *and a positive and* Δ-*differentiable function* $\phi : \mathbf{T} \to \mathbf{R}$ *such that* (3.1) *holds, and*

$$\int_{t_2}^{+\infty} \left[\frac{1}{A(t)} \int_{t_2}^{t} e_{-b/A}\big(t, \sigma(s)\big)\xi^{\lambda}(s)\Phi(s)\Delta s\right]^{1/\lambda} \Delta t = +\infty \tag{3.22}$$

for some constant $t_2 \geq t_0$, *where* $\xi(t) = \int_t^{+\infty} [\frac{e_{-b/A}(s,t)}{A(s)}]^{1/\lambda} \Delta s$, *then* (1.1) *is oscillatory on* $[t_0, +\infty)_{\mathbf{T}}$.

Proof Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1) on $[t_0, +\infty)_{\mathbf{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ for all $t \in [t_1, +\infty)_{\mathbf{T}}$, $t_1 \in [t_0, +\infty)_{\mathbf{T}}$, and then $y(t) > 0$. Analogously, we shall distinguish the following two cases:

$$\text{(I)} \quad y^{\Delta}(t) > 0 \quad \text{for } t \in [t_1, +\infty)_{\mathbf{T}}; \qquad \text{(II)} \quad y^{\Delta}(t) < 0 \quad \text{for } t \in [t_1, +\infty)_{\mathbf{T}}.$$

Case (I) The proof when $y^{\Delta}(t)$ is eventually positive is similar to that of the proof of Theorem 3.1 and it hence is omitted.

Case (II) Since $y(t) > 0$, $y^{\Delta}(t) < 0$ for $t \in [t_1, +\infty)_{\mathbf{T}}$ and $0 < x(t) \leq y(t)$, in view of the definition of $y(t)$ and $0 \leq B(t) < 1$, we have

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} x\big(\tau(t)\big) = \lim_{t\to+\infty} x\big(\delta(t)\big) \geq \frac{1}{2}\lim_{t\to+\infty} y(t) > \frac{1}{3}\lim_{t\to+\infty} y(t),$$

therefore, there exists $t_2 \in [t_1, +\infty)_{\mathbf{T}}$ such that $x(\delta(t)) \geq \frac{1}{3}y(t)$ for all $t \in [t_2, +\infty)_{\mathbf{T}}$.

We proceed as in the proof of Lemma 2.6 to find that $\frac{A(t)|y^{\Delta}(t)|^{\lambda-1}y^{\Delta}(t)}{e_{-b/A}(t,t_0)}$ ($t \in [t_1, +\infty)_{\mathbf{T}}$) is decreasing, and therefore, for all $s \geq t$, $s, t \in [t_1, +\infty)_{\mathbf{T}}$, we have $\frac{A(s)|y^{\Delta}(s)|^{\lambda-1}y^{\Delta}(s)}{e_{-b/A}(s,t_0)} \leq \frac{A(t)|y^{\Delta}(t)|^{\lambda-1}y^{\Delta}(t)}{e_{-b/A}(t,t_0)}$, *i.e.*, $[-y^{\Delta}(s)]^{\lambda} \geq A(t)[-y^{\Delta}(t)]^{\lambda}\frac{e_{-b/A}(s,t)}{A(s)}$, and thus, $y^{\Delta}(s) \leq [A(t)]^{\frac{1}{\lambda}} y^{\Delta}(t) \times [\frac{e_{-b/A}(s,t)}{A(s)}]^{1/\lambda}$, it follows that

$$y(u) - y(t) \leq \big[A(t)\big]^{\frac{1}{\lambda}} y^{\Delta}(t) \int_t^u \left[\frac{e_{-b/A}(s,t)}{A(s)}\right]^{1/\lambda} \Delta s.$$

Now, letting $u \to +\infty$, then we get (in view of Lemma 2.3, we see that $A(t)|y^{\Delta}(t)|^{\lambda-1}y^{\Delta}(t)$ ($t \in [t_1, +\infty)_{\mathbf{T}}$) is decreasing as well)

$$\begin{aligned} y(t) &\geq -\big[A(t)\big]^{\frac{1}{\lambda}} y^{\Delta}(t) \int_t^{+\infty} \left[\frac{e_{-b/A}(s,t)}{A(s)}\right]^{1/\lambda} \Delta s = -\xi(t)\big[A(t)\big]^{\frac{1}{\lambda}} y^{\Delta}(t) \\ &\geq -\xi(t)\big[A(t_1)\big]^{\frac{1}{\lambda}} y^{\Delta}(t_1) = b\xi(t), \end{aligned}$$

where $b_0 = [A(t_1)]^{1/\lambda}[-y^\Delta(t_1)] > 0$ is a constant. Consequently, by (2.2), we find

$$\begin{aligned}-\big[A(t)\varphi\big(y^\Delta(t)\big)\big]^\Delta &\geq b(t)\varphi\big(y^\Delta(t)\big) + \Phi(t)\big[x\big(\delta(t)\big)\big]^\lambda \geq b(t)\varphi\big(y^\Delta(t)\big) + \frac{1}{3^\lambda}\Phi(t)\big[y(t)\big]^\lambda \\ &\geq b(t)\varphi\big(y^\Delta(t)\big) + \frac{b_0^\lambda}{3^\lambda}\Phi(t)\xi^\lambda(t). \end{aligned} \tag{3.23}$$

Define the function $U(t) = A(t)\varphi(y^\Delta(t)) = A(t)|y^\Delta(t)|^{\lambda-1}y^\Delta(t) = -A(t)|y^\Delta(t)|^\lambda$, with (3.23) this yields

$$U^\Delta(t) \leq -\frac{b(t)}{A(t)}U(t) - \frac{b_0^\lambda}{3^\lambda}\Phi(t)\xi^\lambda(t), \quad t \in [t_2, +\infty)_{\mathbf{T}}. \tag{3.24}$$

The inequality in (3.24) is the assumed inequality of [2, Theorem 6.1]. All other assumptions of [2, Theorem 6.1], *e.g.*, $-b/A \in \Re^+$ are satisfied as well. Hence the conclusion of [2, Theorem 6.1] holds, *i.e.*,

$$\begin{aligned} U(t) &\leq U(t_2)e_{-b/A}(t, t_2) - \frac{b_0^\lambda}{3^\lambda}\int_{t_2}^t e_{-b/A}\big(t, \sigma(s)\big)\xi^\lambda(s)\Phi(s)\Delta s \\ &< -\frac{b_0^\lambda}{3^\lambda}\int_{t_2}^t e_{-b/A}\big(t, \sigma(s)\big)\xi^\lambda(s)\Phi(s)\Delta s, \end{aligned}$$

for all $t \in [t_2, +\infty)_{\mathbf{T}}$, *i.e.*, $y^\Delta(t) < -\frac{b_0}{3}[\frac{1}{A(t)}\int_{t_2}^t e_{-b/A}(t, \sigma(s))\xi^\lambda(s)\Phi(s)\Delta s]^{1/\lambda}$, and thus,

$$y(u) < y(t_2) - \frac{b_0}{3}\int_{t_2}^u \left[\frac{1}{A(t)}\int_{t_2}^t e_{-b/A}\big(t, \sigma(s)\big)\xi^\lambda(s)\Phi(s)\Delta s\right]^{1/\lambda}\Delta t \to -\infty \quad \text{as } u \to +\infty,$$

which contradicts with $y(t) > 0$. This completes the proof. □

Using the same method as in the proof of Theorem 3.5, we can now obtain the following results.

Theorem 3.6 *Assume* (H_1)-(H_4), (1.11) *and* (3.22) *hold. If there exists a positive and* Δ-*differentiable function* $\phi : \mathbf{T} \to \mathbf{R}$ *and* $\omega \geq 1$ *such that* (3.10) *holds, then* (1.1) *is oscillatory on* $[t_0, +\infty)_{\mathbf{T}}$.

Theorem 3.7 *Assume* (H_1)-(H_4), (1.11), *and* (3.22) *hold. If there exist functions* $H \in \Omega$, $\zeta_1(t), \zeta_2(t) \in C_{rd}(\mathbf{T}, \mathbf{R})$, *and a positive, nondecreasing, differentiable function* $\phi \in C_{rd}(\mathbf{T}, \mathbf{R}^+)$ *such that* (3.11)-(3.13) *hold, then* (1.1) *is oscillatory on* $[t_0, +\infty)_{\mathbf{T}}$.

Theorem 3.8 *Assume* (H_1)-(H_4), (1.11), *and* (3.22) *hold. If there exist functions* $\zeta_1(t), \zeta_2(t) \in C_{rd}(\mathbf{T}, \mathbf{R})$ *and a constant* $\omega \geq 1$ *such that* (3.19)-(3.21) *hold, then* (1.1) *is oscillatory on* $[t_0, +\infty)_{\mathbf{T}}$.

Remark 3.2 Our results in this paper not only extend and improve some known results, and show some results of [10–17, 21, 22, 25] to be special examples of our results, but also unify the oscillation of the second-order nonlinear delay damped differential equations (1.2) and the second-order nonlinear delay damped difference equations (1.3). The theorems in this paper are new even for the cases $\mathbf{T} = \mathbf{R}$ and $\mathbf{T} = \mathbf{Z}$.

4 Examples

In this section, we give several examples to illustrate our results.

Example 4.1 Consider second-order delay dynamic equation on time scales:

$$\left\{A(t)\varphi\left(\left[x(t)+\left(\frac{1}{\sqrt{2}}-\frac{1}{t}\right)x\bigl(\tau(t)\bigr)\right]^{\Delta}\right)\right\}^{\Delta}+P(t)\varphi\bigl(x\bigl(\delta(t)\bigr)\bigr)=0,\quad t\in\mathbf{T},t\geq t_0,\tag{4.1}$$

here $b(t)\equiv 0$, $Q(t)\equiv 0$, $B(t)=\frac{1}{2}-\frac{1}{2t}$, $F(u)=u$. Pick $A(t)=t^{\frac{2}{3}}$, $\lambda=\frac{5}{3}$, $\tau(t)=\delta(t)=\frac{t}{2}$, $P(t)=\frac{1}{2t}(\frac{2t}{t+2})^{\frac{5}{3}}$, $\mathbf{T}=2^{\mathbf{Z}}$, $t_0=2$, then (4.1) becomes a second-order variable delay 2-difference equation. We have

$$\Phi(t)=LP(t)-\eta Q(t)=\frac{1}{2t}\left(\frac{2t}{t+2\sqrt{2}}\right)^{\frac{5}{3}}>0,$$

$$\int_{t_0}^{t}\left[\frac{e_{-b/A}(s,t_0)}{A(s)}\right]^{\frac{1}{\lambda}}\Delta s=\int_{2}^{t}s^{-\frac{2}{5}}\Delta s=\frac{t^{\frac{3}{5}}-2^{\frac{3}{5}}}{2^{\frac{3}{5}}-1}\to+\infty\quad\text{as } t\to+\infty.$$

Hence, conditions (H_1)-(H_4) and (1.10) are clearly satisfied. Now, in Theorem 3.2, pick $\phi(t)=1$, $\omega=2$. Then for all $t\geq 2$, we can obtain

$$\begin{aligned}
&\limsup_{t\to+\infty}\frac{1}{t^{\omega}}\int_{T_0}^{t}(t-s)^{\omega}\left\{c^{\lambda}\phi(s)\Psi(s)-\frac{(\lambda+1)^{-(\lambda+1)}}{[\phi(s)A(s)]^{\lambda}}\left[\left|\phi^{\Delta}(s)-\frac{\phi(s)b(s)}{A(\sigma(s))}\right|A\bigl(\sigma(s)\bigr)\right]^{\lambda+1}\right\}\Delta s\\
&=\limsup_{t\to+\infty}\frac{1}{t^2}\int_{2}^{t}(t-s)^2\left\{c^{\frac{5}{3}}\frac{1}{2s}\left(\frac{2s}{s+2}\right)^{\frac{5}{3}}\left[1-\left(\frac{1}{2}-\frac{1}{s}\right)\right]^{\frac{5}{3}}\left[\frac{1}{\sqrt{2}}\right]^{\frac{5}{3}}-0\right\}\Delta s\\
&=\left[\frac{c}{\sqrt{2}}\right]^{\frac{5}{3}}\limsup_{t\to+\infty}\frac{1}{t^2}\int_{2}^{t}\frac{(t-s)^2}{2s}\Delta s\\
&=\left[\frac{c}{\sqrt{2}}\right]^{\frac{5}{3}}\limsup_{t\to+\infty}\left\{\frac{1}{2}\bigl[\log_2(t)-1\bigr]+\frac{1}{2t^2}\left(\frac{t^2-2^2}{2^2-1}\right)-\frac{1}{t}(t-2)\right\}=+\infty.
\end{aligned}$$

Therefore, by Theorem 3.2, (4.1) is oscillatory.

Example 4.2 Consider second-order nonlinear variable delay dynamic equation on time scales $\mathbf{T}$:

$$\bigl[t^{\frac{2}{5}}x^{\Delta}(t)\bigr]^{\Delta}+t^{-\frac{13}{5}}x^{\Delta}(t)+t^{-\frac{11}{10}}x\left(\frac{t}{2}\right)=0,\quad t\in\mathbf{T}=2^{\mathbf{Z}},t\geq t_0:=2,\tag{4.2}$$

here $A(t)=t^{2/5}$, $B(t)\equiv 0$, $b(t)=t^{-13/5}$, $P(t)=t^{-11/10}$, $Q(t)\equiv 0$, $F(u)=u$, $\delta(t)=t/2$, $\lambda=1$. Similarly, it is easy to see that conditions (H_1)-(H_4) and (1.10) are satisfied. Now, take $\omega=2$, $\phi(t)=1$, then we have

$$\begin{aligned}
&\limsup_{t\to+\infty}\frac{1}{t^{\omega}}\int_{T_0}^{t}(t-s)^{\omega}\left\{c^{\lambda}\phi(s)\Psi(s)-\frac{(\lambda+1)^{-(\lambda+1)}}{[\phi(s)A(s)]^{\lambda}}\left[\left|\phi^{\Delta}(s)-\frac{\phi(s)b(s)}{A(\sigma(s))}\right|A\bigl(\sigma(s)\bigr)\right]^{\lambda+1}\right\}\Delta s\\
&=\limsup_{t\to+\infty}\frac{1}{t^2}\int_{2}^{t}(t-s)^2\left\{c\frac{2^{-2}}{s^{11/10}}-\frac{2^{-2}}{s^{2/5}}\bigl(s^{-13/5}\bigr)^2\right\}\Delta s\\
&=\frac{1}{4}\limsup_{t\to+\infty}\frac{1}{t^2}\int_{2}^{t}(t-s)^2\left(\frac{c}{s^{11/10}}-\frac{1}{s^{28/5}}\right)\Delta s<+\infty,
\end{aligned}$$

which implies that (3.10) does not hold. Therefore, Theorem 3.1 and Theorem 3.2 cannot be applied to (4.2), and one can easily see that the results in [10–17, 21, 22, 25] cannot be applied in (4.2).

Next, we will apply Theorem 3.4 and it remains to satisfy the conditions (3.19)-(3.21). Since

$$\begin{aligned}
&\limsup_{t\to+\infty}\frac{1}{t^\omega}\int_u^t (t-s)^\omega\Psi(s)\Delta s\\
&\quad=\limsup_{t\to+\infty}\frac{1}{t^2}\int_u^t (t-s)^2\frac{2^{-2}}{s^{11/10}}\Delta s=\frac{1}{4}\limsup_{t\to+\infty}\frac{1}{t^2}\int_u^t\bigl(t^2s^{-11/10}-2ts^{-1/10}+s^{9/10}\bigr)\Delta s\\
&\quad=\frac{1}{4}\frac{1}{u^{1/10}-1}\geq\frac{1}{4u^{1/10}}=\xi_1(u),\\
&\limsup_{t\to+\infty}\frac{1}{t^\omega}\int_u^t\frac{(t-s)^\omega}{A^\lambda(s)}\biggl[\frac{\omega A(\sigma(s))}{t-s}+b(s)\biggr]^{\lambda+1}\Delta s\\
&\quad=\limsup_{t\to+\infty}\frac{1}{t^2}\int_u^t\frac{1}{s^{2/5}}\bigl[2(2s)^{2/5}+(t-s)s^{-13/5}\bigr]^2\Delta s\\
&\quad=\limsup_{t\to+\infty}\frac{1}{t^2}\int_u^t\bigl[2^{\frac{14}{5}}s^{\frac{2}{5}}+2^{\frac{12}{5}}ts^{-\frac{13}{5}}-2^{\frac{12}{5}}s^{-\frac{8}{5}}+t^2s^{-\frac{28}{5}}-2ts^{-\frac{23}{5}}+s^{-\frac{18}{5}}\bigr]\Delta s\\
&\quad=\frac{u^{-23/5}}{1-u^{-23/5}}\leq\frac{u^{-23/5}}{1-2^{-23/5}}=\xi_2(u),
\end{aligned}$$

we have

$$\xi_1\bigl(\sigma(s)\bigr)-\theta\xi_2\bigl(\sigma(s)\bigr)=\frac{1}{4(2s)^{1/10}}-\theta\frac{(2s)^{-23/5}}{1-2^{-23/5}}=\frac{k_1}{s^{1/10}}-\frac{k_2}{s^{23/5}},$$

where $k_1=\frac{1}{4\cdot 2^{1/10}}$, $k_2=\frac{2^{-23/5}}{1-2^{-23/5}}\theta$. We conclude

$$\begin{aligned}
&\liminf_{t\to+\infty}\frac{1}{t^\omega}\int_{T_0}^t\frac{(t-s)^\omega A(s)[\xi_1(\sigma(s))-\theta\xi_2(\sigma(s))]_+^{(\lambda+1)/\lambda}}{[A(\sigma(s))]^{(\lambda+1)/\lambda}}\Delta s\\
&\quad=\liminf_{t\to+\infty}\frac{1}{t^2}\int_2^t\frac{(t-s)^2s^{2/5}(\frac{k_1}{s^{1/10}}-\frac{k_2}{s^{23/5}})^2}{[(2s)^{2/5}]^2}\Delta s\\
&\quad=\frac{1}{2^{4/5}}\liminf_{t\to+\infty}\frac{1}{t^2}\int_2^t(t-s)^2\biggl(\frac{k_1^2}{s^{3/5}}-2\frac{k_1k_2}{s^{51/10}}+\frac{k_2^2}{s^{48/5}}\biggr)\Delta s=+\infty.
\end{aligned}$$

So, (3.19)-(3.21) are satisfied as well. Altogether, by Theorem 3.4, we find that (4.2) is oscillatory.

Remark 4.1 One can easily see that the recent results cannot be applied in (1.1) or (4.1) and (4.2), so our results are new ones.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The first author discovered the topic, offered the main ideas for the proof of the paper, and carried out writing this article. The second author gave some helpful suggestions for writing the paper and checked the proof of the paper. All authors read and approved the final manuscript.

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